Improving approximations of the Mohr-Coulomb failure envelope for triaxial stress conditions

by Jesse Amundsen with Sarah Johnson as the faculty sponsor
Department of Physics and Geology, Northern Kentucky University

Abstract

In order to more accurately find the Mohr-Coulomb failure envelope, differential calculus is used to explicitly find the angle of internal friction and the cohesion of a fictional material. An iterative method for solving for the failure envelope is explored using MATLAB. To illustrate the explicit method a problem containing only two stress samples is completed. Eventually, the iterative approximation method is revealed to be a more preferable and realistic approach when solving for the failure envelope due to its scalability, speed and acceptable accuracy. To test the program, six sample datasets are processed and the results are used to further explore the uses and properties of the Mohr-Coulomb failure envelope.

Background

The Mohr-Coulomb failure envelope describes how materials behave under stress. Most materials follow the Mohr-Coulomb failure envelope, for at least part of their failure envelope, though brittle materials such as rock and concrete are more accurately described by it. The failure envelope itself is described by the relationship of the shear strength of the material and the amount of normal stress being applied. To obtain a failure envelope for a material Mohr's Circles are often plotted and then the failure envelope is determined as the line most tangent to the available Mohr's Circles (Figure 01). The goal of this project is to develop a method that removes the need for graphical approximation of the failure envelope and to provide a method that can arrive at the most accurate failure envelope possible when given large amounts of variable, non-ideal data.

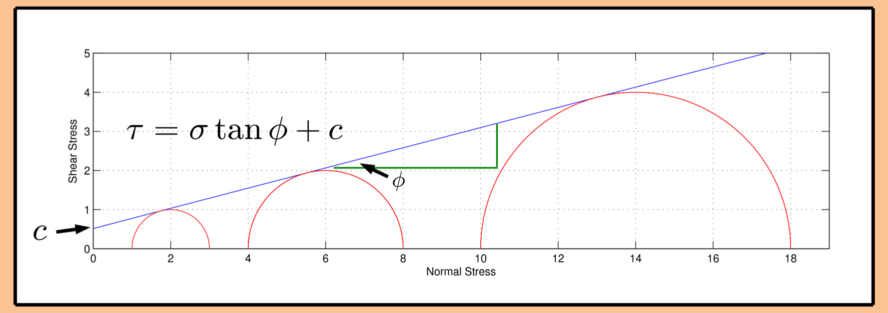

Figure 01: An idealized Mohr-Coulomb failure envelope (blue line) for three Mohr's Circles (shown in red). The y-intercept of the failure envelope identifies the cohesion (c) of the material and the slope of the line (φ) indicates the angle of internal friction.

Methods

First, differential calculus is used to find the exact line tangent to two individual Mohr's Circles. This is referred to as the explicit method. The second method takes advantages of the ability of computers to do simple mathematical operations quickly and uses iterative approximation in MATLAB and is referred to as the iterative method. The ultimate goal is to generate a method that is accurate, fast, and can perform approximations for large volumes of data.

Explicit Solution

To evaluate the line tangent to two Mohr's circles, two continuous functions are constructed by using two samples of stress data. This is done by recognizing the relationship between the principal stresses and Mohr's Circle.

$$y = \sqrt{[\frac{1}{2}(\sigma_1 - \sigma_3)]^2 - [x - (\sigma_3 + \frac{1}{2}(\sigma_1 - \sigma_3))]^2}$$

After constructing continuous functions representing each Mohr's Circle (referred to as *f(x)* and *g(x)*), they are each differentiated. In this example, σ_3 is 1 and 4 and σ_1 is 3 and 8. Two new x-values are introduced which represent where the point of tangency is on each Mohr's circle, x_f and x_g. The functions are set equal to one another and either x_f or x_g is solved for. This shows where the slope of each circle is identical.

$$f'(x_f) = g'(x_g) \qquad \frac{-(x_f - 2)}{\sqrt{-x_f^2 + 4x_f - 3}} = \frac{-(x_g - 6)}{\sqrt{-x_g^2 + 12x_g - 32}}$$

$$x_f = \frac{1}{2}(x_g - 2) \qquad x_g = 2(x_f + 1)$$

$$x_f = -\frac{1}{2}(x_g - 10) \qquad x_g = -2(x_f - 5)$$

$$f'(x_f) = m = \frac{g(x_g) - f(x_f)}{x_g - x_f} = \frac{-(x_f - 2)}{\sqrt{-x_f^2 + 4x_f - 3}}$$

$$\frac{-(x_f - 2)}{\sqrt{-x_f^2 + 4x_f - 3}} = \frac{\sqrt{4 - ((2(x_f + 1)) - 6)^2} - \sqrt{1 - (x_f - 2)^2}}{2(x_f + 1) - x_f}$$

$$\boxed{x_f = \frac{7}{4}}$$

The value of x_f represents the x-value on the Mohr's Circle $f(x_f)$ where the slope of the line tangent to it at that point is also tangent to the Mohr's circle $g(x_g)$. Using this point we can find the slope at $f'(x_f)$ and a corresponding y-value at $f(x_f)$.

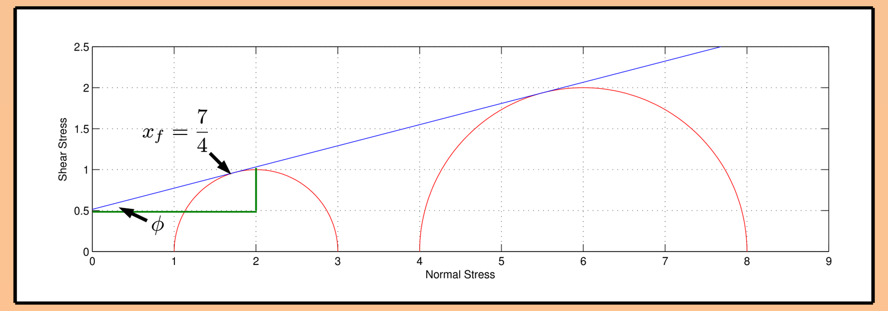

Figure 02: Two Mohr's Circles where σ_3 is 1 and 4 and σ_1 is 3 and 8. Indicated is where x=7/4. This shows where the point of tangency is on the first Mohr's circle. At this point, the line tangent to the first Mohr's circle is also tangent to the second Mohr's circle.

To find the angle of internal friction (φ) trigonometry is used (Figure 02). The cohesion (c) of the material is the y-intercept.

$$y_c - \frac{\sqrt{15}}{4} = \frac{\sqrt{15}}{15}(x_c - \frac{7}{4}) \qquad \sin\phi = \frac{f(2) - c}{2} = \frac{\frac{4\sqrt{15}}{15} - \frac{2\sqrt{15}}{15}}{2}$$

$$c = \frac{-\sqrt{15}}{15}\frac{7}{4} + \frac{\sqrt{15}}{4} = \boxed{\frac{2\sqrt{15}}{15}} \qquad \phi = \arcsin\left(\frac{2\frac{\sqrt{15}}{15}}{2}\right) = \boxed{14.9632^\circ}$$

Iterative Solution

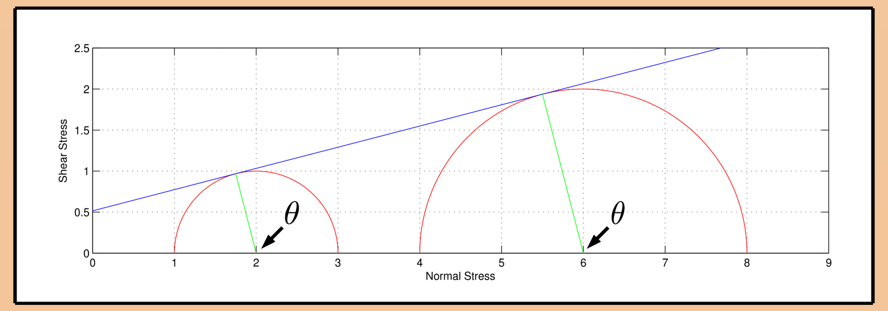

Figure 03: Using the same Mohr's Circles from the explicit example this diagram indicates that the angle formed by a line from the centroid of the circle to the point of tangency will be identical when the optimal failure envelope is found. The y-intercept will be at its maximum at this point.

The key to this method is to recognize that the angle θ in each Mohr's Circle will always be equal when a line is drawn from the centroid of the circle to the point of tangency on each Mohr's Circle (Figure 03). When constraints have been put in place to force it so that the second stress pair will always have a larger differential stress and both σ_3 and σ_1 values that are greater than those in the first stress pair, the y-intercept of the set of points of tangency that are on both half-circles for the same angle θ will be greatest when the most optimal pair is found (Figure 04). Using this knowledge, it is then possible to divide the circle into increments and test each set of dividing points to find the maximum y-intercept. This is most easily performed in a mathematical programming environment such as MATLAB.

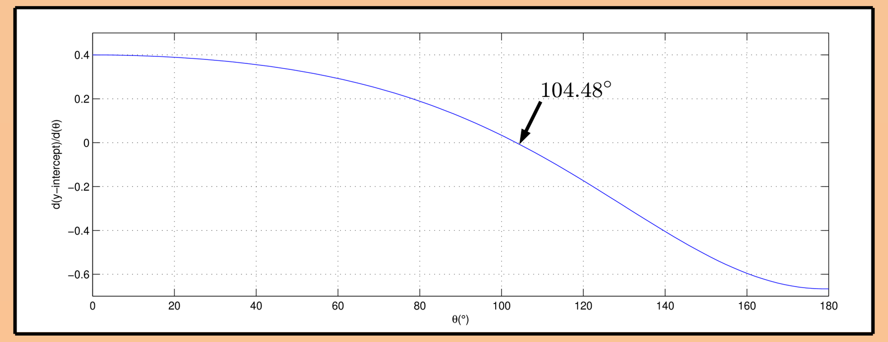

Figure 04: The relationship between the change in the y-intercept and the angle θ is graphically represented to show that it crosses the x-axis at approximately 104°. This is the most optimal angle of θ in both Mohr's Circles (in this case for those in the examples). A graph such as this can be generated for any two Mohr's Circles.

Comparison of the Solutions

The explicit method is time consuming and difficult compared to the iterative method. The iterative method allows for the processing of large amounts of stress data in a short amount of time and provides an adequate amount of accuracy to be confident in the determined values for the cohesion and angle of internal friction. Due to the reality that the most optimal failure envelope will be found when larger amounts of stress samples are taken and all of the combinations of failure envelopes are averaged to find the best fit failure envelope, the scalability of the iterative approximation method makes it the clear choice for nearly all situations.

Data Analysis

To test the implementation of the iterative solution six sample datasets were run through the solver to generate cohesion (c), and angle of internal friction (φ) values. For this analysis, an aggregate of data on the Dunham dolomite, Solenhofen limestone, Mizuho trachyte, Shirahama sandstone, KTB amphibolite, and an unnamed marble was used from Al-Ajmi (2006). Only data points where $\sigma_2 = \sigma_3$ were used.

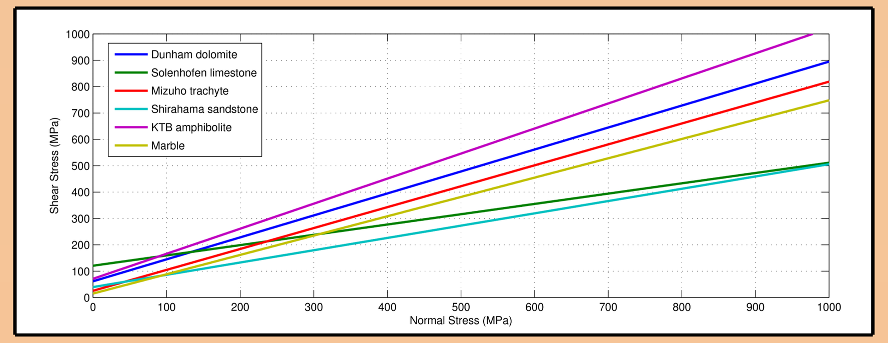

Figure 05: Mohr-Coulomb failure envelopes for the Dunham dolomite, Solenhofen limestone, Mizuho trachyte, Shirahama sandstone, KTB amphibolite, and an unnamed marble approximated by the iterative method.

	Dunham dolomite	Solenhofen limestone	Mizuho trachyte	Shirahama sandstone	KTB amphibolite	marble
c (MPa)	61.53383	120.66708	25.54780	39.57955	71.05321	15.04539
φ (°)	56.44541	23.00197	52.49628	27.79034	71.81827	47.14121

Table 01: Approximated values using the Mohr-Coulomb failure envelope to find the cohesion and angle of internal friction for the six sample data sets seen in Figure 01. Values for the Dunham dolomite, Solenhofen limestone, and KTB amphibolite are close matches with approximations made by Colmenares and Zoback (2002).

The iterative approximation method produces similar results to those made by Colmenares and Zoback (2002) for the Dunham dolomite, Solenhofen limestone, and KTB amphibolite. Comparing values from the Mogi-Coulomb criterion using equations from Al-Ajmi (2006) to relate the Mogi parameters *a* and *b* to *c* and φ showed that both predicted similar values for the Solenhofen limestone and Shirahama sandstone.

Obtaining Data

Test data is obtained by applying stress to a sample of the material in question until the sample fails or fractures (Figure 05). Numerous samples can be run at varying confining stress values to generate a set of triaxial data.

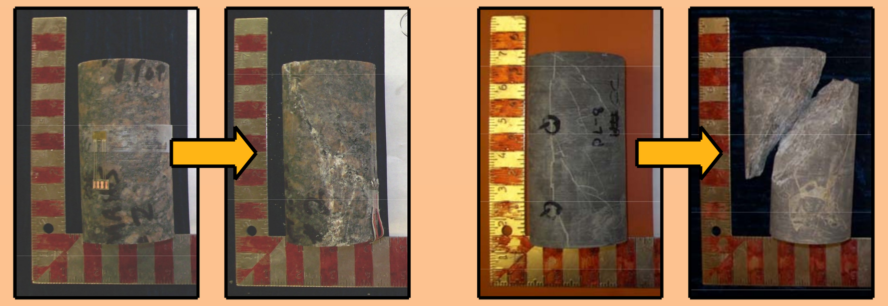

Figure 05: Two rock samples used for triaxial tests showing the sample prior to testing and after (courtesy of Kleinfelder, Inc.). The first sample is a Granite and the second is a Micrite. Both show fracturing indicating that they had exceeded their failure envelope.

Discussion

Since the Mohr-Coulomb failure criterion ignores the effects of the intermediate principal stress (σ_2), Colmenares and Zoback (2002) note that it makes more accurate predictions for materials that do not have σ_2 dependent failure. It is important to consider that the Mohr-Coulomb failure envelope only represents a portion of the failure envelope which often curves downward as it approaches σ_n=0 from the right and lowers its slope for high values of σ_n (Figure 07). The Mohr-Coulomb failure envelope most accurately describes the region between these two graphical features.

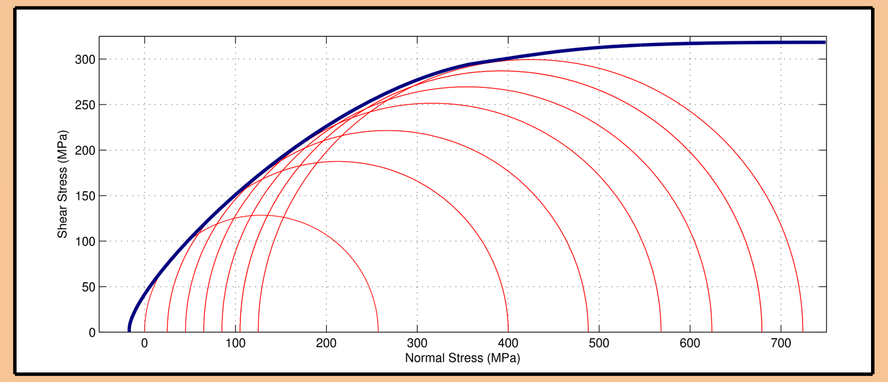

Figure 07: A hypothetical failure envelope for the Dunham dolomite showing its tendency to curve downward as it approaches tension and the leveling off that occurs under extremely high stress. The Mohr-Coulomb failure envelope most accurately predicts the region between these two features.

The cohesion values predicted by the Mohr-Coulomb envelope were typically higher than expected which is a result of the slight curving downward that the failure envelope sometimes exhibits as it approaches tension (Figure 07). The iterative approximation method of the Mohr-Coulomb failure envelope is a fast, easy way to obtain cohesion and angle of internal friction values for a material that fractures with minimal respect to the intermediate principal stress and that has triaxial test data with varying confining stress.

Selected References

[1] "Brittle fracture of rock" by Evert Hoek and edited by K.G. Stagg and O.C. Zienkiewicz. Chapter 4 of *Rock Mechanics in Engineering Science*, Chapter 4, 1968.

[2] "A statistical evaluation of intact rock failure criteria constrained by polyaxial test data for five different rocks" by L.B. Colmenares and M.D. Zoback. Published in the International Journal of Rock Mechanics & Mining Sciences, Vol. 39, pp. 695-729, 2002.

[3] "Wellbore stability analysis based on a new true-triaxial failure criterion" by Adel Al-Ajmi. Submission for Ph.D. candidacy, KTH Land and Water Resources Engineering, 2006.

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